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RADIO INTERFEROMETRY BY LUNAR REFLECTIONS

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ABSTRACT

We discuss the use of the Moon as a passive reflector for radio interferometry experiments with baselines on the order of the radius of the orbit of the Moon. Because of the extreme loss in the lunar interferometer path, only exceptionally strong point sources are candidates for this kind of interferometry. As the Moon is far from an ideal reflector due to the irregular surface, the data processing necessary to optimize the fringe detectability is rather complicated. We describe the optimum processing procedure and compute the signal-to-noise ratio for these processors. Two specific applications are discussed, one being the spatial resolution of Jupiter bursts, the other the resolution of the H_2O maser at 22 GHz in Orion. We show that both experiments are definitely possible with existing (Arecibo) or currently planned telescopes (Green Bank).
Subject headings: interferometry — radio sources: general

I. INTRODUCTION

The use of a passive reflector to produce interference fringes has been applied in various forms in radio observations for some time. The sea (or "cliff") interferometer is an early example of the application of this method to resolve radio sources (for a description and references, see Thompson, Moran, and Swenson 1986, chap. 1). Making use of the Moon as a passive reflector in an extremely high resolution interferometer was discussed in the late 1960s at MIT Lincoln Laboratory in connection with the extensive radio and radiometric investigations of the Moon then taking place there. The implementation of the experiment, first suggested for OH emissions from W3, was abandoned then because it was felt that the signal-to-noise ratio would be too marginal for fringe detection. Since then the idea has been revived by A. E. E. Rogers (1986, private communication) for possible studies of the extremely strong H_2O emission from Orion (Reid and Moran 1988). One of us (J. A. P.) has also proposed to use this technique to try to spatially resolve S-type Jupiter bursts (Carr, Desch, and Alexander 1983). The Moon interferometer idea has also been suggested by Soviet scientists (Artyukh and Shishov 1982), who gave a relatively crude analysis of the problem upon which we wish to expand. So far the scheme has not found any successful applications and can best be described as a technique seeking a problem. It appears, however, that for the two applications mentioned above there is a realistic hope of positive results. As the method is of marginal sensitivity, we shall discuss in this paper the various problems which arise and how they may be minimized for optimum sensitivity.

If the Moon were a perfectly reflecting smooth sphere, it would be quite straightforward to assess the performance of such an interferometer. However, the lunar material is far from a perfect reflector. The surface may be considered as a nearly lossless dielectric with a dielectric constant of ~ 2.7, varying somewhat with the observing frequency. The reflectivity will change with angle of incidence and, furthermore, will depend on the polarization of the wave measured relative to the limb of the Moon. More serious complications arise because the lunar surface is not smooth. Instead of the return coming from a Fresnel zone centered on the geometric reflection point, the actual reflections arise from a large number of specular points associated with individual irregularities. The returns will be distributed over a sizable fraction of the lunar surface and will also be spread in time delay. This reduces the correlation between adjacent frequency components in the signal spectrum and means that the effective bandwidth for coherent processing is limited. Further problems arise because the Moon is in rotation due to physical and apparent librations. This small rotational motion brings about differential Doppler effects which cause the output signal of the interferometer to fade both in amplitude and phase. This fading limits the time span available for coherent integration.

It is the purpose of this paper to discuss in some detail the principles of such interferometry experiments and to establish the lower limits of the flux required of a point source in order to detect fringes as a function of the frequency of observation. We begin by considering the simplest case of the Moon modeled as a smooth dielectric sphere. We then proceed to regard it as a rough sphere capable of reflecting at a range of different time delays. Finally we take into account the effect of librations which make the reflected signal vary with time and set limits on the time span for coherent integration which depend on time delay. In the final discussion we establish the lower flux limits for point sources as a function of the parameters of the experiment, and in particular consider the Orion H_2O emission and the Jupiter *S*-bursts. The latter experiment will be considered in greater detail in a separate paper describing some actual observations now being carried out at the Arecibo Observatory.

II. THE SMOOTH MOON

In this section we shall first compute the effective interferometer baseline, assuming the Moon to be a perfect sphere. We then compute the complex cross correlation of the direct and the reflected signals, and the effective signal-to-noise ratio of the detected fringes for given system temperatures of the two antenna systems used. The effective system noise of the antenna viewing the reflection point on the Moon may be dominated by the thermal emission from the Moon, and since the two antennas are likely to be of different effective apertures it is necessary to decide which of the two antennas to use for the Moon reflected signal. For the dielectric sphere we determine the dependence of fringe visibility on angle of incidence and polarization.

Figure 1 shows the geometry of the two rays used. In Appendix A we derive an expression for the effective baseline and conclude that it is equal to AB in Figure 1. An approximate expression for this baseline is

$$D = R_m \sin \phi - \frac{2a}{1 + \cos \phi}, \tag{1}$$

where R_m is the distance to the center of the Moon, a is the radius of the Moon, and ϕ is the angular distance between Moon and source.

If the angular separation between the Moon and the source exceeds $\sim 2°$, the effective baseline will be greater than Earth's diameter. The longest achievable baseline using the lunar interferometry method equals the Moon's orbital radius (3.8×10^5 km), when the source and the Moon are separated by 90°.

Assume that a plane wave of flux density S_0 is incident on the Earth-Moon system. The wave arriving along the indirect path will have a flux density given by

$$S_m = |\rho_m(\phi)|^2 S_0 a^2/4R_m^2 = \alpha^2 S_0 , \tag{2}$$

where ρ_m is the lunar reflection coefficient. This means that the field in the indirectly arriving wave is reduced in the ratio $\alpha = |\rho_m(\phi)| a/2R_m$ or by a factor of about $|\rho_m(\phi)| 2.26 \times 10^{-3}$ as compared with the direct wave. For ρ_m we may use either of the two Fresnel reflectivities:

$$\rho_{\parallel}(\phi) = \frac{\epsilon \sin \phi/2 - \sqrt{\epsilon - \cos^2 \phi/2}}{\epsilon \sin \phi/2 + \sqrt{\epsilon - \cos^2 \phi/2}}, \qquad \rho_{\perp}(\phi) = \frac{\sin \phi/2 - \sqrt{\epsilon - \cos^2 \phi/2}}{\sin \phi/2 + \sqrt{\epsilon - \cos^2 \phi/2}}, \tag{3}$$

where $\epsilon \approx 2.7$. The $\parallel$ and the $\perp$ indices refer to the plane containing the source, the lunar reflection point and the observer. Figure 2 shows the Fresnel coefficients as functions of ϕ and of the effective baseline length. Clearly, the strength of the reflected signal will be enhanced for geometries with grazing incidence.

The complex correlation of the signals received in the two antennas, one pointed at the Moon and the other at the source, will be reduced by the factor α. Let the antenna signals from the direct and the indirect paths be denoted by $e_1(t)$ and $\alpha e_2(t)$, respectively. The voltage $e_2(t)$ is the signal one would obtain from antenna 2 if it were placed at point A1 in Figure 1 and pointed directly at the source. We avoid the single- or double-sideband complications introduced by superheterodyne mixing by dealing with the antenna signals directly. Our procedure, therefore, corresponds to single sideband reception. The total signals from the two antennas,

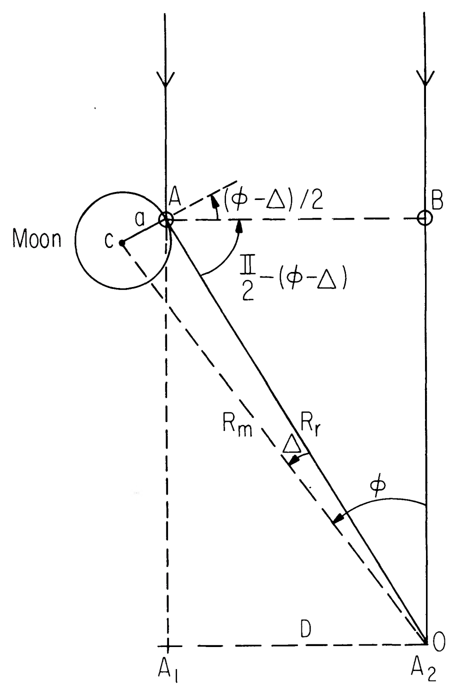

FIG. 1

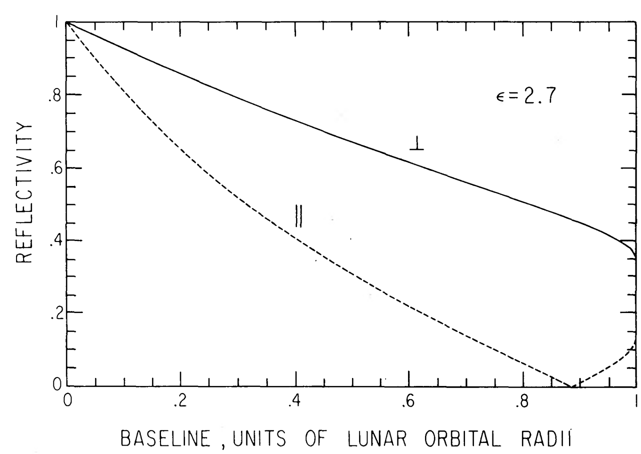

FIG. 2

FIG. 1.—The geometry used to derive the effective baseline

FIG. 2.—The two Fresnel reflection coefficients plotted as a function of the effective baseline

including additive noise of whatever origin, will be represented as

$$\text{direct} \quad f_1(t) = e_1(t) + n_1(t) , \qquad \text{reflected } f_2(t) = \alpha e_2(t) + n_2(t) .$$

For a point source at infinity, and assuming identical antenna impedances, we have

$$e_2(t) = \sqrt{\frac{A_2}{A_1}}\, e_1(t - \theta_m) , \tag{4}$$

where $\theta_m \approx R_m(1 - \cos \phi)/c$ is the extra delay in the indirect path, and where A_1 and A_2 are the two antenna apertures.

The complex covariance of $f_1(t)$ and $f_2(t)$ at time delay θ_m becomes

$$\langle r_c(\theta_m)\rangle = \alpha\langle e_1(t)e_2^*(t + \theta_m)\rangle = \langle f_1(t)f_2^*(t + \theta_m)\rangle , \tag{5}$$

where $\langle\cdots\rangle$ denotes averaging over the *signal* ensemble, and where we have made use of the fact that the two noise terms are uncorrelated. In practice we estimate the ensemble average from a finite (coherent) time average over the interval of observation T_0:

$$r_c(\theta_m) = \frac{1}{T_0}\int_0^{T_0} r(t, \theta_m)dt = \frac{1}{T_0}\int_0^{T_0} f_1(t)f_2^*(t + \theta_m)dt \tag{6}$$

which is an unbiased estimator of $\alpha\langle e_1(t)e_2^*(t + \theta_m)\rangle$.

By virtue of the central-limit theorem (see, e.g., Papoulis 1965), $r_c(\theta_m)$ is a complex Gaussian random variable with a mean given by equation (5) and with a variance:

$$\langle |\Delta r_c(\theta_m)|^2\rangle = \frac{1}{T_0^2}\int_0^{T_0}\int_0^{T_0} \langle |f_1|^2\rangle\langle |f_2|^2\rangle \rho_1(t - t')\rho_2(t - t')dt\,dt' , \tag{7}$$

where ρ_1 and ρ_2 are the normalized autocorrelation functions of f_1 and f_2, respectively. If the power spectra of f_1 and f_2 are both limited to a bandwidth B, and both have constant power density within that band, then

$$\rho_1(t - t')\rho_2(t - t') = \left\{\frac{\sin [\pi B(t - t')]}{\pi B(t - t')}\right\}^2 . \tag{8}$$

In this particular case we obtain

$$\langle |\Delta r_c(\theta_m)|^2\rangle = \frac{1}{BT_0}\langle |f_1|^2\rangle\langle |f_2|^2\rangle . \tag{9}$$

The (coherent) ratio of the signal to noise becomes

$$\left(\frac{s_i}{n_0}\right)_c = \frac{\langle r_c(\theta_m)\rangle}{\sqrt{\langle |\Delta r_c(\theta_m)|^2\rangle}} = \frac{\alpha\langle e_1(t)e_2^*(t + \theta_m)\rangle\sqrt{BT_0}}{\sqrt{\langle |f_1|^2\rangle\langle |f_2|^2\rangle}} . \tag{10}$$

If the flux density of the source in the observed polarization is $S_0/2$ (the flux density of a source is usually given as the sum of the flux density of both polarizations), and the effective aperture of the antenna pointed at the source is A_1, and that pointed at the Moon A_2, the received power per polarization in the two antennas will be

$$W_{s1} = BA_1 S_0/2 = \langle |e_1|^2\rangle/Z , \qquad W_{s2} = \alpha^2 BA_2 S_0/2 = \alpha^2\langle |e_2|^2\rangle/Z , \tag{11}$$

where Z is the impedance assumed the same for both antennas. The additive noise powers in the two antennas are

$$W_{n1} = BkT_1 = \langle |n_1|^2\rangle/Z , \qquad W_{n2} = BkT_2 = \langle |n_2|^2\rangle/Z , \tag{12}$$

where k is a Boltzmann's constant, $1.38 \times 10^{-23} J/K$.

For the signal-to-noise ratio we obtain

$$\frac{s_i}{n_0} = \frac{\alpha S_0\sqrt{A_1 A_2}}{\sqrt{(A_1 S_0 + 2kT_1)(\alpha^2 A_2 S_0 + 2kT_2)}}\sqrt{BT_0} . \tag{13}$$

In the Moon path we always expect the signal power to be much less than the noise power, i.e., $\alpha^2 A_2 S_0 \ll 2kT_2$, leading to the following simplification:

$$\frac{s_i}{n_0} = \frac{aS_0\sqrt{A_1 A_2}}{\sqrt{(2kT_1 + A_1 S_0)2kT_2}}\sqrt{BT_0} . \tag{14}$$

If, furthermore, the signal power in the antenna pointed directly at the source is well above the noise level, i.e., $2kT_1 \ll A_1 S_0$, we obtain

$$\frac{s_i}{n_0} = \alpha\sqrt{\frac{S_0 A_2}{2kT_2}}\sqrt{BT_0} , \tag{15}$$

Hence, as long as the conditions stated above apply, the signal-to-noise ratio depends only on the ratio A_2/T_2 of the antenna pointed at the Moon. Clearly, in this case the antenna with the highest value of this ratio should always be used for the Moon path.

In order to obtain an idea about the viability of this method for the two examples quoted above, consider first some representative numbers for the Jupiter burst experiment. Jupiter bursts have not yet been resolved with Earth-based interferometers, and the determination of the size of the emitting region remains an important clue as to their physical origin. In an experiment currently underway at the Arecibo Observatory, we are using a four-element cross Yagi at 25 MHz to observe S-bursts from Jupiter. The 305 m dish is used to observe the lunar reflection. If the burst has a strength of pM Jy we have the following parameters:

$$S_0 = p \times 10^{-20}\ \mathrm{W\ m^{-2}\ Hz^{-1}}\,, \qquad A_1 = 115\ \mathrm{m}^2\,,$$
$$A_2 = 2 \times 10^4\ \mathrm{m}^2\,, \qquad T_1 = T_2 = 2 \times 10^4\ \mathrm{K}\,,$$
$$\alpha = 2 \times 10^{-3};$$

T_1 and T_2 are the galactic background temperatures at 25 MHz; the contribution from the thermal noise of the Moon is completely negligible at this frequency; α was computed assuming a lunar reflection coefficient $\rho_m = 0.71$ corresponding to $\phi = 15°$ and $\epsilon = 2.7$. In this case we have

$$\frac{s_i}{n_0} \approx 0.08 \frac{p}{\sqrt{1+2p}} \sqrt{BT_0}$$

where we estimate the added system temperature from Jupiter bursts to be 4×10^4 pK assuming an antenna gain of 0.04 K Jy^{-1} for the Yagi antenna. We see that for weak bursts the signal-to-noise scales as p but for very strong bursts only as $p^{1/2}$. The S-bursts normally come in quasi-periodic pulse trains lasting 1 or more seconds. The duration of an individual burst is typically 10 ms. For a train of S-bursts lasting one second a signal should be detectable at the $\sim 6\ \sigma$ level, assuming a bandwidth of 10 kHz and a mean strength of 1 MJy. A typical "storm" may consist of several hundred such burst groups so the signal-to-noise can be further improved by averaging the correlation from many groups incoherently. Clearly, there are complications with the coherent integration if the source moves during a burst train. Nevertheless, as p is often in the range 1–10, we believe the experiment may give positive results. We shall show below that the depth and the fading of the Moon may not be so detrimental at this frequency and for such short bursts.

Consider next the second example, that of the H_2O maser in Orion. For an experiment in Green Bank, say, one could use the 140 foot dish as the source-directed antenna and the planned Green Bank large steerable dish for the Moon observations. Assuming gains of 0.1 K Jy^{-1} and 1 K Jy^{-1} for the 140 foot and 300 foot antennas, respectively, the following parameters are expected to apply:

$$S_0 = 1\ \mathrm{MJy}\,, \quad B = 10^5\ \mathrm{Hz}\,, \quad A_1 = 300\ \mathrm{m}^2\,, \quad A_2 = 2800\ \mathrm{m}^2\,, \quad T_1 = 1 \times 10^5\ \mathrm{K}\,, \quad T_2 = 250\ \mathrm{K}\,, \quad \alpha = 1.4 \times 10^{-3}\,.$$

The system temperature T_1 is assumed to be due to the H_2O maser which dominates the sky background and receiver temperatures; T_2 is the approximate temperature of the Moon; and α was computed assuming $\phi = 45°$ and $\epsilon = 2.7$. In this case we obtain:

$$\frac{s_i}{n_0} = 60\sqrt{T_0}\,.$$

The delay spreading and reflectivity variation with time may be detrimental for this observation at 22 GHz, but, as we shall see below, detection may still be expected.

III. THE ROUGH MOON: SPREAD IN DELAY

In § II the Moon-reflected signal was taken to come from a single Fresnel zone about the geometric reflection point, and the reflected signal $\alpha e_2(t)$ was considered not to be distorted. Irregularities in the lunar surface cause this Fresnel zone to break up into numerous smaller contributions spread over large parts of the surface, and, hence, spread in time delay. We shall defer to the next section the discussion of the effects of variation with time of this delay spread echo. The reflected signal now takes the form:

$$e_2'(t) = \alpha \int_0^\infty e_2(t-\theta) m_B(\theta) d\theta\,, \tag{16}$$

where $m_B(\theta)$ is a random impulse response, smoothed by the bandwidth of the signal, B, which we shall take to be a zero mean complex Gaussian process. It is the (truncated) Fourier transform of the normalized frequency response $M(f)$ of the Moon channel:

$$m_B(\theta) = \int_{-B/2}^{B/2} M(f) e^{2\pi i f \theta} df\,. \tag{17}$$

In the cases considered in the previous section there was complete coherence over the whole frequency band, and the peak in the cross correlation versus time delay was determined by the shape of the autocorrelation of the incoming signal alone. For a rough Moon, the correlation width in frequency is often much less than the bandwidth of the incoming signal, and the shape of the cross correlation function is determined by the coherence function of the Moon rather than by the autocorrelation of the signal. We shall describe the random impulse response of the Moon by the following statistical properties (Hagfors 1961):

$$\langle m_B(\theta)\rangle_m = 0\ , \quad \frac{1}{B}\langle | m_B(\theta)|^2\rangle_m = \int_{-B/2}^{B/2} \mathscr{R}(\Delta f)\, e^{2\pi i \Delta f \theta}\, d\Delta f = p_B(\theta)\ , \quad \langle m_B(\theta) m_B^*(\theta')\rangle_m = \langle | m_B(\theta)|^2\rangle_m \frac{\sin \pi B(\theta - \theta')}{\pi B(\theta - \theta')}\ ,$$
$$\int_{-\infty}^{\infty} p_B(\theta)\, d\theta = 1\ . \tag{18}$$

We have inserted an index m on the statistical averaging symbol to emphasize that this random process is different from and independent of the random process which creates the incoming signal. The frequency correlation, or coherence function, $\mathscr{R}(\Delta f)$ is determined by the coherence of adjacent frequencies in the Moon path. The impulse response $m(\theta)$ and $\mathscr{R}(\Delta f)$ are controlled jointly by the properties of the lunar surface and of the sharpness of the beam of the antenna pointed toward the Moon. The relative density of reflected power with time delay is $p_B(\theta)$.

Because of the spread in delay we must now form our estimate of the cross correlation of the two signals on the basis of a number of separate and independent estimates, one for each delay. Suppose we form a coherent estimate of the cross correlation of f_1 and f_2 as we did in § II (see eq. [6]), but now for each additional delay $\delta\theta_k$ separately:

$$r_c(\theta_m + \delta\theta_k) = \frac{1}{T_0}\int_0^{T_0} r(\tau, \theta_m + \delta\theta_k) d\tau\ , \tag{19}$$

where θ_m is the time delay to the specular point on the Moon. If we take the mean value of this over the *signal* ensemble, we obtain

$$\langle r_c(\theta_m + \delta\theta_k)\rangle = \alpha\langle e_1 e_2^*\rangle m^*(\theta_m + \delta\theta_k)\Delta\theta\ . \tag{20}$$

The $m(\theta_m + \delta\theta_k)$ is the value of the random impulse response in the vicinity of the delay $\theta_m + \delta\theta_k$ and $\Delta\theta = 1/B$ is the delay resolution interval. As m is a zero mean random process, further coherent averaging of r_c over the Moon ensemble is futile (i.e., $\langle\langle r_c\rangle\rangle_m = 0$). However, information on $\langle e_1 e_2^*\rangle$ is contained in $| r_c|^2$ because

$$\langle\langle | r_c |^2\rangle\rangle_m = \sigma_m^2 \langle | m_k |^2\rangle_m + \sigma_0^2\ , \tag{21}$$

where

$$\sigma_m^2 = \alpha^2 | \langle e_1 e_2^*\rangle |^2 \Delta\theta^2\ , \qquad \sigma_0^2 = \langle | f_1 |^2\rangle\langle | f_2 |^2\rangle / BT_0\ .$$

The unbiased estimator of $| \langle e_1 e_2^*\rangle |^2$ for the time delay $\delta\theta_k$, therefore, is

$$(| \langle e_1 e_2^*\rangle |^2)_{Ek} = \frac{r_c r_c^* - \sigma_0^2}{\alpha^2 \langle | m_k |^2\rangle_m \Delta\theta^2}\ . \tag{22}$$

The variance of this estimator of $| \langle e_1 e_2^*\rangle |^2$ when the correlation is zero, i.e., the background " noise," is given by

$$\Delta_k^2 = \frac{\langle (r_c r_c^* - \sigma_0^2)^2\rangle}{(\alpha^2 \langle | m_k |^2\rangle_m \Delta\theta^2)^2} = \frac{\sigma_0^4}{(\alpha^2 \langle | m_k |^2\rangle_m \Delta\theta^2)^2}\ . \tag{23}$$

If one were to make an estimate of the actual value of the correlation the variance ought to be computed with the correlation set equal to the actual value, different from zero. Here we shall only be concerned with the detection of a correlation, and, therefore, use the variance with zero correlation.

The weighted average estimate, taking all the individual delay estimates into account, becomes

$$(| \langle e_1 e_2^*\rangle |^2)_E = \sum_{k=0}^{M} \beta_k (| \langle e_1 e_2^*\rangle |^2)_{Ek}\ . \tag{24}$$

The overall estimate with the least variance is obtained by choosing the weights proportional to the inverse of the individual estimates of the variances, in other words:

$$\beta_k \sum_{l=0}^{M} \frac{1}{\Delta_l^2} = \frac{1}{\Delta_k^2} = \frac{1}{\sigma_0^4} (\alpha^2 \langle | m_k |^2\rangle_m \Delta\theta^2)^2\ . \tag{25}$$

With this choice of weights the overall variance becomes

$$\Delta^2 = \left(\sum_{k=0}^{M} \frac{1}{\Delta_k^2}\right)^{-1} = \left(\frac{\sigma_0}{\alpha\Delta\theta}\right)^4 \left(\sum_{k=0}^{M} \langle | m_k |^2\rangle_m^2\right)^{-1}\ . \tag{26}$$

The final signal-to-noise ratio corresponding to the coherent one as given in equation (10) becomes

$$\frac{s_i}{n_0} = \left(\frac{s_i}{n_0}\right)_c \left(\Delta\theta^4 \sum_{k=0}^{M} \langle | m_k |^2\rangle_m^2\right)^{1/4}\ . \tag{27}$$

Rather than further discuss this unrealistic model, where the coherent integration interval equals the total observing time, we shall proceed to the more realistic case where the coherent integration interval is determined by the time variation of the lunar echo due to the librations.

IV. THE ROUGH MOON: TIME VARIATION AND SPREAD IN DELAY

In the previous section we considered the Moon-reflected signal as resulting from an ensemble of different realization of the lunar surface. In reality, because of the variation types of librations of the Moon the response of the Moon path will pass from one realization to the next with time, and the impulse response of the Moon path is a time-varying random process. The averaging over the Moon ensemble, denoted by $\langle \cdots \rangle_m$ in § III, can be estimated as a time average. Instead of equation (16) we have

$$e'_2(t) = \alpha \int_0^{\infty} e_2(t-\theta) m_B(\theta; t) d\theta , \tag{28}$$

where $m_B(\theta; t)$ is a random, smoothed, time-varying impulse response. It is the (truncated) Fourier transform of the normalized frequency response $M(f; t)$ of the Moon path, and equation (18) is replaced by

$$m_B(\theta; t) = \int_{-B/2}^{B/2} M(f; t) e^{2\pi i f \theta} df . \tag{29}$$

For the random, time-varying impulse response of the Moon we shall assume the following further statistical properties in addition to those of equations (19) (Hagfors 1961):

$$\begin{aligned} \frac{1}{B} \langle m_B(\theta; t) m_B^*(\theta; t+\sigma) \rangle_m &= p_B(\theta) \rho_B(\theta, \sigma) \\ &= \int_{-B/2}^{B/2} \mathscr{R}(\Delta f; \sigma)\, e^{2\pi i \Delta f \theta} d\Delta f , \\ w_B(\theta, f) &= \int_{-\infty}^{\infty} \rho_B(\theta, \sigma)\, e^{-2\pi i (f\sigma)} d\sigma , \\ \int_{-\infty}^{\infty} w_B(\theta, f) df &= 1 . \end{aligned} \tag{30}$$

The function $\mathscr{R}(\Delta f; \sigma)$ is the double autocorrelation function of the time-varying frequency response $M(f; t)$ relating the response at frequencies separated by Δf and times separated by σ, and $w_B(\theta, f)$ is the normalized power spectrum of the return at range θ.

In this case of a time varying reflector we cannot form r_c by integrating over the complete observing period. Instead we have to integrate the cross products $r(t, \theta_m + \delta\theta_k)$ with a time-varying weight function $h(t, \tau)$ which depends on the coherence span of the signal at the delay under consideration. We shall omit the reference to this time delay $\theta_k = \theta_m + \delta\theta_k$ or just indicate the time delay by an index k for ease of writing, but emphasize that the weights h depend on the time delay. We shall try to determine these weights in such a way that the ultimate variance of the estimates are minimized:

$$r_c(t) = \int h(t, \tau) r(\tau) d\tau . \tag{31}$$

The signal ensemble mean value of this estimate is

$$\langle r_c(t) \rangle = \alpha \langle e_1 e_2^* \rangle \Delta\theta \int h(t, \tau) m^*(\tau) d\tau . \tag{32}$$

As in the previous section $m^*(\theta, \tau)$ is taken to be a zero mean complex Gaussian process. The mean over signal and Moon ensembles of $| r_c |^2$ is

$$\begin{aligned} \langle\!\langle | r_c |^2 \rangle\!\rangle_m &= \iint h(t, \tau) h^*(t, \tau') \langle\!\langle r(\tau) r^*(\tau') \rangle\!\rangle_m d\tau\, d\tau' \\ &= \sigma_m^2 \langle | m |^2 \rangle_m \iint h(t, \tau) h^*(t, \tau') \rho_B(\tau - \tau') d\tau\, d\tau' + \sigma_0^2 T_0 \int | h(t, \tau) |^2 d\tau . \end{aligned} \tag{33}$$

In computing the best weights by minimizing the variance of the estimates of $| r_c |^2$ for the various delays it is possible to determine weight functions $h(t, \tau)$ which vary with time t in order to properly account for the end effects in the time interval T_0. This could be done by developing $\rho_B(\tau' - \tau)$ in a Karhunen-Loève expansion (see, e.g., van Trees 1968) over this time interval. For simplicity we shall be satisfied here with a suboptimum estimator which is good as long as the coherence span of the Moon channel is short compared to T_0. We therefore put

$$h(t, \tau) = h(t - \tau) . \tag{34}$$

With these simplifications the variance of the estimator of $| \langle e_1 e_2^* \rangle |^2$ (eq. [23]) when the signal correlation is absent, i.e., the background " noise " (corresponding to eq. [24]), is given by

$$\Delta_k^2 = \frac{\sigma_0^4 \int_0^{T_0} \int_0^{T_0} dt\, dt' \iint d\tau\, d\tau' h_k(t-\tau) h_k^*(t'-\tau) h_k^*(t-\tau') h_k(t'-\tau')}{[\alpha^2 \Delta\theta^2 \langle | m_k |^2 \rangle_m \iint h_k(t-\tau) h_k^*(t-\tau') \rho_{Bk}(\tau - \tau') d\tau\, d\tau']^2} . \tag{35}$$

In order to determine the minimum we introduce the power spectrum of the Moon channel fluctuations at delay k and for a delay smoothing interval $\Delta\theta$ as defined above, and the frequency spectrum of $h(t)$ by $H(f)$ and find that we must make

$$\frac{\int |H(f)|^4 df}{|\int |H(f)|^2 w_{Bk}(f) df|^2} \tag{36}$$

a minimum. By the Schwarz inequality it follows that we must choose

$$|H(f)|^2 = w_{Bk}(f) \tag{37}$$

With this choice of weights we find for Δ_k^2:

$$\Delta_k^2 = \frac{\sigma_0^2 T_0}{(\alpha^2 \Delta\theta^2 \langle |m_k|^2 \rangle_m)^2 \int w_{Bk}^2(f) df} . \tag{38}$$

With the weights for each delay in inverse proportion to the variance, as in § III, we finally obtain for the overall variance:

$$\Delta^2 = \left(\sum_{k=0}^{M} \frac{1}{\Delta_k^2}\right)^{-1} = \left(\frac{\sigma_0}{\alpha \Delta\theta}\right)^4 \left[\sum_{k=0}^{M} (\langle |m_k|^2 \rangle_m)^2 \frac{1}{T_0} \int w_{Bk}^2(f) df\right]^{-1} . \tag{39}$$

The final signal-to-noise ratio, replacing the coherent one of equation (10), becomes

$$\frac{s_i}{n_0} = \left(\frac{s_i}{n_0}\right)_c \left[\Delta\theta^4 \sum_{k=0}^{M} \langle |m_k|^2 \rangle_m^2 \frac{1}{T_0} \int w_{Bk}^2(f) df\right]^{1/4} . \tag{40}$$

This expression for the signal-to-noise ratio has the same form as for the coherent case, equation (10), except that the bandwidth B and the observing time T_0 must be replaced, respectively, by

$$B' = \sqrt{B b_{coh}} , \qquad T_0' = \sqrt{T_0 t_{coh}} . \tag{41}$$

It is also instructive to write the ultimate signal-to-noise ratio in the form:

$$\frac{s_i}{n_0} = \frac{\alpha \langle e_1 e_2^* \rangle \sqrt{b_{coh} t_{coh}}}{\sqrt{\langle |f_1|^2 \rangle \langle |f_2|^2 \rangle}} \sqrt[4]{\frac{B T_0}{b_{coh} t_{coh}}} . \tag{42}$$

In this form it is clear that the signal-to-noise ratio is first computed for the coherence bandwidth and coherence time available, and then there is an incoherent summation over the number of such coherence intervals both in frequency and in time.

The product of the coherence bandwidth b_{coh} and the coherence time t_{coh} now has been given a precise definition in terms of the statistical properties of the Moon expressed in terms of the target scattering function:

$$b_{coh} t_{coh} = \sum_{k=0}^{M} \left(\frac{\langle |m_k|^2 \rangle_m}{B}\right)^2 \Delta\theta \int w_{Bk}^2(f) df . \tag{43}$$

For a realistic assessment of lunar interferometer experiments all we have to do is to substitute the statistical properties of the lunar surface as a reflector of electromagnetic waves into the formulae derived. These properties are discussed in the next section.

V. STATISTICAL PROPERTIES OF THE ROUGH MOON

The functions describing the statistical properties of the Moon have been well studied in the case of backscattering (Evans and Hagfors 1968, chap. 5), but not for forward scattering as required here. It is, therefore, necessary to extrapolate as best we can from the backscatter data to the situation at hand. If we denote the time delay by θ, the corresponding backscatter time delay θ_0 (the maximum is 11.6 ms), expressed in terms of the offset angle ϕ introduced in § II, becomes

$$\theta_0 = \theta / \sin(\phi/2) . \tag{44}$$

The backscattering properties change with the wavelength of observation. For long wavelengths the Moon appears rather smooth, and most of the scattered energy comes from the immediate vicinity of the geometric reflection point. For shorter wavelengths the Moon becomes gradually rougher, and the quasi-specular returns give way to a more diffuse scattering. We shall assume, without very strong justification, that the scattering law at oblique angles at a frequency f is identical to the backscattering law at a frequency f_0, properly scaled to account for the increased reflectivity at oblique angles, given by

$$f_0 = f \sin(\phi/2) . \tag{45}$$

The Doppler width of the scattered signal can be related to the corresponding Doppler width of the backscattered signal through the relation:

$$\delta f_D(\theta; f) = \delta f_{0D}\left[\frac{\theta}{\sin(\phi/2)}; f \sin\left(\frac{\phi}{2}\right)\right] , \tag{46}$$

where $\delta f_D(\theta; f)$ is the Doppler width of the Moon reflected signal at delay θ and center frequency f, whereas $\delta f_{0D}(\theta_0; f_0)$ is the corresponding backscatter Doppler width at backscatter delay θ_0 and backscatter center frequency f_0. Figure 3 shows examples of

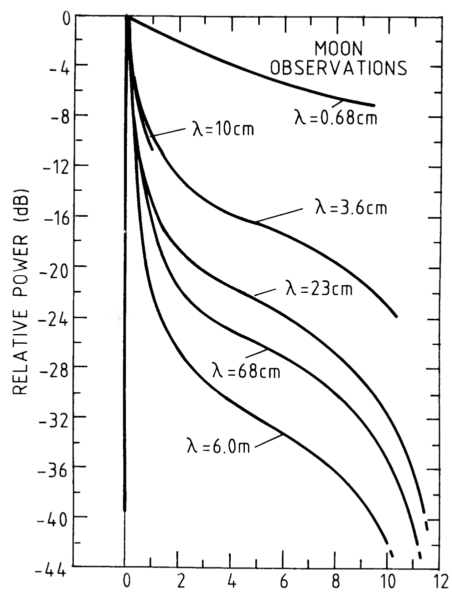

FIG. 3

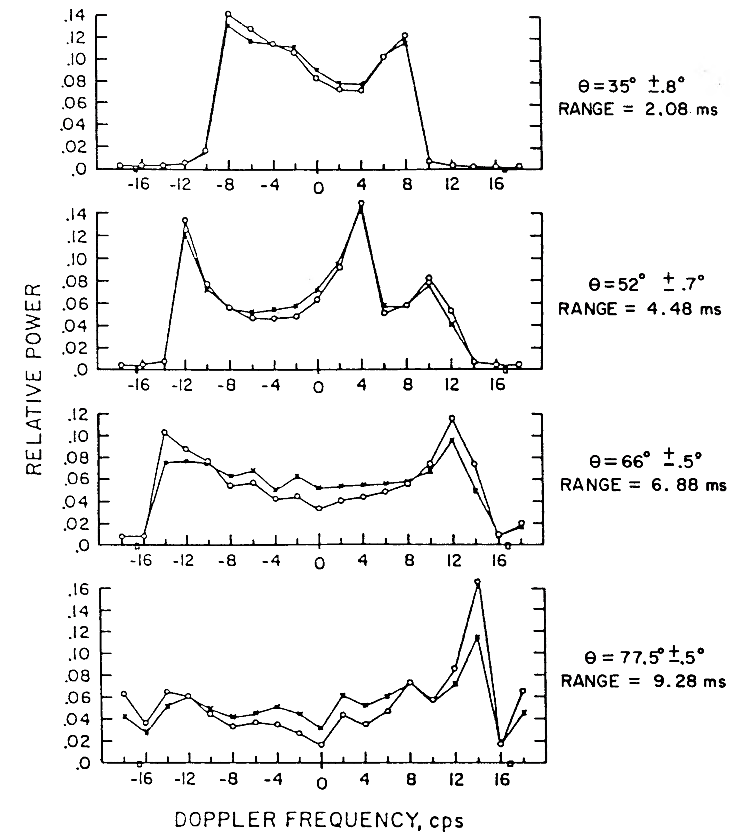

FIG. 4

FIG. 3.—The power vs. delay plots for backscatter from the Moon at a number of different wavelengths of observation

FIG. 4.—Backscatter frequency spectra for various delays for a wavelength of observation of 23 cm, for two orthogonal linear polarizations

backscattered power versus time delay for a number of different radar frequencies. Figure 4 gives an example of power spectra as a function of time delay for backscatter at one particular radar frequency. The analytic expression for $\delta f_{0D}(\theta_0; f_0)$ is

$$\delta f_{0D}(\theta_0; f_0) = \delta f_{0\,\max} \sqrt{2\frac{\theta_0}{\Theta_0} - \left(\frac{\theta_0}{\Theta_0}\right)^2} \tag{47}$$

where Θ_0 is the maximum backscatter time delay of 11.6 ms and where $\delta f_{0\,\max}$ is the limb-to-limb Doppler width.

VI. DISCUSSION

Let us first discuss one procedure which may be followed in order best to detect interference fringes in lunar interferometer experiments and then as examples consider the prospects for the two specific experiments mentioned in § I.

Assuming that the two antennas are appropriately pointed,the first step is to record the two signals f_1 and f_2 in some form without serious distortion. The two complex signals must then be cross multiplied in order to form the quantities we have denoted by $r(t, \theta)$ and used to form r_c in equations (6), (19), and (31). The delay θ must be allowed to assume a large number of values, also values different from those corresponding to the time delay of any point on the Moon. These " off the Moon " delays are necessary in order to establish a noise baseline; see equation (23). The complex cross products are then Fourier analyzed over the complete observing interval T_0 for all possible time delays, and the Fourier amplitudes are squared. This allows the convolution implied in equation (33) to be carried out as a straight multiplication of spectra. From this we obtain a two-dimensional array of numbers corresponding to time delay θ and frequency offset f. From the assumed properties of the Moon, and from the geometry and frequency of observation the function $w_B(\theta, f)$ (the target function) is constructed, and the cross products of this assumed array and the observed array are computed, and the products are summed over all the cross products. This procedure is carried out with $w_B(\theta, f)$ placed on the observed array at the known position of the Moon, and at a large number of positions where it is off the position of the Moon. The mean of the " off Moon " sums is then subtracted from the " on Moon " sum, and, if the latter value is significantly higher than the rms value of the " off Moon " sums, a detection of fringes may be said to have occurred.

In the Jupiter case, we shall assume a frequency of observation of 25 MHz, and an offset angle ϕ of 10°. The equivalent backscattering frequency is then 2.18 MHz. In Figure 3 the lowest backscattering power versus delay curve is for 50 MHz, and, if we are to approximate the initial spike of this curve with an exponential, the time constant would be $\sim 150\ \mu$s. We do not know how to extrapolate this to 2.18 MHz, and to be conservative, let us assume the same value of 150 μs there. Because of the forward-scatter geometry this translates to a time constant in our " target scattering function " of 13 μs. A typical value of the limb-to-limb Doppler width of a backscatter signal at 25 MHz would be 0.6 Hz. Because of the forward-scatter geometry this will be reduced to 50 mHz. At a delay of 150 μs, equation (47) tells us that the Doppler width is only 16% of the maximum, or 8 mHz. From this we conclude that the coherence bandwidth b_{coh} is on the order of 80 kHz and that the coherence time t_{coh} is on the order of 125 s. As we expect burst bandwidths on the order of 10 kHz and burst durations of 100 ms (see § II), it is clear that we can ignore all the complications of delay spread and time variation, and compute the signal-to-noise ratio for a single burst coherently as we did in § II. This experiment is, therefore, very promising.

In the Orion H_2O maser at 22 GHz the equivalent backscatter frequency at an offset angle of 45° is 8.42 GHz. At this frequency the equivalent time constant for an exponential power delay curve is $\sim 310\ \mu$s. With the obliquity factor at 45° this is reduced to 119 μs (τ_0). A typical backscattering Doppler width at 22 GHz is 515 Hz ($\delta f_{0\,max}$), which is reduced by the obliquity factor to 197 Hz (δf_{max}). We now assume that an exponential approximation to the initial returned power versus delay with a "time constant" of 119 μs (τ_0), and rectangular power spectra of width given by equation (47) with $\delta f_{0\,max}$ replaced by 197 Hz (δf_{max}), Θ_0 by 4.4 ms (Θ) and θ_0 by θ. The rectangular approximation will give a suboptimum and hence conservative solution. For this particular case we find

$$b_{coh}\, t_{coh} = \frac{1}{2\sqrt{\pi}\,\tau_0\,\delta f_{max}} \sqrt{\frac{\Theta}{\tau_0}} \, .$$

With the numerical values given above and in § II for this case we find for the signal-to-noise ratio:

$$\frac{s_i}{n_0} = 51.8 \sqrt[4]{T_0}$$

which is still quite respectable.

Finally, we should add that the use of reduced bit multiplications and possible other nonoptimum schemes will further reduce the signal-to-noise ratio. Nevertheless, both the specific experiments discussed above look promising and should be attempted. If there are other extremely strong sources which might be candidates for lunar interferometry, we have established the complete data processing procedure and formulas to determine the ultimate signal-to-noise ratio in each case.

We are indebted to A. E. E. Rogers for calling our attention to the possible application of this technique to the Orion maser and to P. Nicholson for many helpful discussions. NAIC operates the Arecibo Observatory under cooperative agreement with the National Science Foundation. One of us (T. H.) has been the recipient of an Alexander von Humboldt Foundation Award during the course of the work.

APPENDIX A

THE EFFECTIVE BASELINE

From the triangle ACO in Figure 1 we have

$$\frac{a}{\sin \Delta} = \frac{R_m}{\cos [(\phi - \Delta)/2]} = \frac{R_r}{\cos [(\phi - \Delta)/2]} \, . \tag{A1}$$

It follows that

$$R_r = a \frac{\cos [(\phi + \Delta)/2]}{\sin \Delta} \tag{A2}$$

and the difference in pathlength becomes

$$\Delta R = AO - BO = R_r[1 - \cos (\phi - \Delta)] \, . \tag{A3}$$

The angle Δ can be determined from the transcendental equation:

$$\frac{a}{R_m} \cos [(\phi - \Delta)/2] = \sin \Delta \, . \tag{A4}$$

Substitution of $\Delta = \phi_0 + \epsilon$, where $\sin \phi_0 = a/R_m$ allows us to solve equation (A4) by expanding to first order in ϵ:

$$\epsilon = \frac{2(a/R_m) \sin^2 [(\phi - \phi_0)/4]}{\cos \phi_0 - (a/2R_m) \sin [(\phi - \phi_0)/2]} \tag{A5}$$

From equation (A5) it is possible to show that $\epsilon \ll \phi_0$ for $\phi < 80°$. Since ϕ will probably be less than 80° for any practical lunar interferometry experiment, we can neglect ϵ and expand equation (A3) to first order in ϕ_0 to obtain:

$$\Delta R = R_m(1 - \cos \phi) - 2a \tan (\phi/2) \, . \tag{A6}$$

The effective baseline D (AB in Fig. 1) can be determined from ΔR by $D = (d/d\phi)(\Delta R)$. Using equation (A6) we obtain equation (1). For more accurate and general results it appears to be necessary to solve equation (A4) numerically.

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